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Approximation orders of formal Laurent series by Oppenheim rational functions

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Abstract

We study formal Laurent series which are better approximated by their Oppenheim convergents. We calculate the Hausdorff dimensions of sets of Laurent series which have given polynomial or exponential approximation orders. Such approximations are faster than the approximation of typical Laurent series (with respect to the Haar measure).

Keywords: Finite field; Laurent series; Oppenheim convergents; Approximation; Hausdorff dimension

1. Statements of results

Let $q \geqslant 2$ be an integer and $\mathbb{F}_q$ be a finite field of q elements. Let $\mathscr{L} = \mathbb{F}_q((z^{-1}))$ denote the field of all formal Laurent series $A = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate z, with coefficients c_n all lying in the field $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denotes the ring of polynomials in z with coefficients in $\mathbb{F}_q$.

For the above formal Laurent series A, we may assume that $c_v \neq 0$. Then the integer $v = v(A)$ is called the *order* of A. The *norm* (or *valuation*) of A is defined to be $||A|| = q^{-v(A)}$. With the convention $v(0) = +\infty$ and $||0|| = 0$, we have the following:

(i) $||A|| \geqslant 0$ with $||A|| = 0$ iff $A = 0$;
(ii) $||AB|| = ||A|| \cdot ||B||$;

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(iii) $||\alpha A + \beta B|| \leqslant \max(||A||, ||B||)$ $(\forall \alpha, \beta \in \mathbb{F}_q)$;
(iv) $\alpha \neq 0,\ \beta \neq 0,\ ||A|| \neq ||B|| \Rightarrow ||\alpha A + \beta B|| = \max(||A||, ||B||)$.

In other words, $||\cdot||$ is a non-Archimedean norm of the field $\mathscr{L}$. It is well known that $\mathscr{L}$ is a complete metric space under the metric ρ defined by $\rho(A,B) = ||A - B||$. See [5, Chapter 5] for more information on the normed field $\mathscr{L}$. See also [1,4,6,9].

For $A = \sum_{n=v}^{\infty} c_n z^{-n} \in \mathscr{L}$, let $[A] = \sum_{v \leqslant n \leqslant 0} c_n z^{-n} \in \mathbb{F}_q[z]$. We call $[A]$ the *integral part* of A. It is evident that the integer $-v(A) := -v$ is equal to the degree $\deg [A]$ of the polynomial $[A]$ of z.

The Oppenheim expansions of Laurent series were introduced by Knopfmacher and Knopfmacher [7,8]. Let us now recall the definition.

Let $\{r_n\}_{n \geqslant 1}$ and $\{s_n\}_{n \geqslant 1}$ be two sequences of non-zero polynomials over $\mathbb{F}_q$ satisfying the following hypothesis:

$$\text{(H)} \quad v\left(\frac{r_n}{s_n}\right) \leqslant 2, \ \text{i.e.} \ \deg s_n - \deg r_n \leqslant 2 \quad (\forall n \geqslant 1).$$

This condition is natural, as we shall see later, for the algorithm described below to be effective.

Given $A \in \mathscr{L}$, denote that $a_0 = [A]$. Then we define recursively a finite or an infinite sequence of formal Laurent series $\{A_n\}$ associated to A. Define $A_1 = A - a_0$. Suppose A_n $(n \geqslant 1)$ is defined. If $A_n \neq 0$, then let $a_n = [\frac{1}{A_n}]$ and define

$$A_{n+1} = \left(A_n - \frac{1}{a_n}\right)\frac{s_n(a_n)}{r_n(a_n)}, \tag{1}$$

where $s_n(a_n)$ and $r_n(a_n)$ denote the composition of polynomials. If $A_n = 0$, this recursive process stops. We call $\{a_n\}$ the *digits* of A. It was shown [7,8] that the above algorithm leads to a finite or an infinite series which converges to A (relative to ρ). This series (see (2)) is called the *Oppenheim expansion* of Laurent series of A.

Theorem A (Knopfmacher and Knopfmacher [7,8]). *Every $x \in \mathscr{L}$ has a finite or an infinite convergent (relative to ρ) expansion of the form*

$$x = a_0(x) + \frac{1}{a_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(a_1(x)) \cdots r_n(a_n(x))}{s_1(a_1(x)) \cdots s_n(a_n(x))} \frac{1}{a_{n+1}(x)}, \tag{2}$$

where $a_n(x) \in \mathbb{F}[z]$, $a_0(x) = [x]$, and $\deg a_1(x) \geqslant 1$, for any $n \geqslant 1$,

$$\deg a_{n+1}(x) \geqslant 2 \deg a_n(x) + 1 + \deg r_n(a_n(x)) - \deg s_n(a_n(x)). \tag{3}$$

The expansion is unique for x subject to the proceeding admissibility condition (3) *on the digit $a_n(x)$.*

Let I denote the valuation ideal $z^{-1}\mathbb{F}_q[[z^{-1}]]$ in the ring of formal power series $\mathbb{F}_q[[z^{-1}]]$. It consists of all formal series $\sum_{n=1}^{\infty} c_n z^{-n}$. For any $n \geqslant 1$, define the map

$$T_n 0 = 0, \quad T_n x = \left(x - \frac{1}{[1/x]}\right)\frac{s_n([1/x])}{r_n([1/x])} \quad \text{for } x \in I \backslash \{0\}.$$

It may be checked that T_n maps I into I iff hypothesis (H) is satisfied, and that

$$a_n(x) = a(T_{n-1}\circ\cdots\circ T_2\circ T_1 x) \quad \text{with} \quad a(x) = [1/x].$$

Here are some special cases which were extensively studied:
Lüroth expansion: $s_n(z) = z(z-1),\ r_n(z) = 1$;
Engel expansion: $s_n(z) = z,\ r_n(z) = 1$;
Sylvester expansion: $s_n(z) = 1,\ r_n(z) = 1$;
Cantor infinite product: $s_n(z) = z,\ r_n(z) = z+1$.

The ideal I is compact because it can be identified with $\prod_{n=1}^{\infty} \mathbb{F}_q$ (the metric ρ restricted on I is exactly the usual ultra-metric on $\prod_{n=1}^{\infty} \mathbb{F}_q$). A natural measure on I is the normalized Haar measure on $\prod_{n=1}^{\infty} \mathbb{F}_q$, which we denote by $\mathbf{P}$. We now consider the approximations of formal Laurent series by rational functions. For any $x\in I$, consider the partial sums

$$\omega_n(x) = \frac{1}{a_1(x)} + \sum_{j=2}^{n} \frac{r_1(a_1(x))\cdots r_{j-1}(a_{j-1}(x))}{s_1(a_1(x))\cdots s_{j-1}(a_{j-1}(x))} \frac{1}{a_j(x)},$$

which are called *Oppenheim convergents* of x. The following metric theorem was proved in [3] for a large class of Oppenheim expansion under a stronger hypothesis than (H).

Theorem B (Fan and Wu [3]). *Suppose the following hypothesis is satisfied*:

(HL) $\deg s_n - \deg r_n = L$ *for any* $n\geqslant 1$

where $L\leqslant 2$ *is a fixed integer. Then*

(i) *If* $L=2$, *then for* $\mathbf{P}$-*almost all* $x\in I$,
$$\lim_{n\to\infty} \frac{1}{n}\log_q \|x-\omega_n(x)\| = -\frac{2q}{q-1}.$$

(ii) *If* $L=1$, *then for* $\mathbf{P}$-*almost all* $x\in I$,
$$\lim_{n\to\infty} \frac{1}{n^2}\log_q \|x-\omega_n(x)\| = -\frac{q}{2(q-1)}.$$

(iii) *If* $L\leqslant 0$, *then for* $\mathbf{P}$-*almost all* $x\in I$,
$$\lim_{n\to\infty} \frac{1}{(2-L)^n}\log_q \|x-\omega_n(x)\| = -\frac{1}{1-L}G(x),$$
where $0<\frac{2-L}{1-L}\leqslant G(x)<\infty$ *for* $\mathbf{P}$-*almost all* $x\in I$.

Roughly speaking, when $L=2$, the last theorem means that for $\mathbf{P}$-almost all $x\in I$, we have $\|x-\omega_n(x)\|\approx q^{-\frac{2q}{q-1}n}$. We could say that x is approximated by its convergents $\omega_n(x)$ with linear order $\frac{2q}{q-1}n$. We would like to know which Laurent series can be approximated with polynomial orders αn^β $(\alpha > 0, \beta > 1)$ or exponential orders $\xi\tau^n$ $(\xi > 0, \tau > 1)$. Similar questions arise when $L = 1$ or $L \leqslant 0$. We answer these questions by the following theorems.

Theorem 1. *Suppose $L \leqslant 0$. For any $\tau > 2 - L$ and $\xi > 0$, we have*

$$\dim\left\{x \in I\colon \frac{1}{\tau^n}\log_q \|x - \omega_n(x)\| \to -\xi\right\} = \frac{1}{L + \tau - 1}.$$

Theorem 2. *Suppose $L = 1$. For any $\beta > 1$ and $\alpha > 0$, we have*

$$\dim\left\{x \in I\colon \frac{1}{n^\beta}\log_q \|x - \omega(x)\| \to -\alpha\right\} = 1.$$

For any $\tau > 1$ and $\xi > 0$, we have

$$\dim\left\{x \in I\colon \frac{1}{\tau^n}\log_q \|x - \omega_n(x)\| \to -\xi\right\} = \frac{1}{\tau}.$$

Theorem 3. *Suppose $L = 2$. For any $\beta > 1$ and $\alpha > 0$, we have*

$$\dim\left\{x \in I\colon \frac{1}{n^\beta}\log_q \|x - \omega(x)\| \to -\alpha\right\} = \frac{1}{2}.$$

For any $\tau > 1$ and $\xi > 0$, we have

$$\dim\left\{x \in I\colon \frac{1}{\tau^n}\log_q \|x - \omega_n(x)\| \to -\xi\right\} = \frac{1}{\tau + 1}.$$

Let $E(\tau, \xi)$ be the set of $x \in I$ such that $\frac{1}{\tau^n}\log_q \|x - \omega_n(x)\| \to -\xi$. See Fig. 1, which compares the dimensions of $E(\tau, \xi)$ following the values of L. Remark that the dimensions, which are independent of ξ, have a common formula $\frac{1}{L+\tau-1}$. But the domains of definition are different.

2. Preliminaries

From now on, we always assume that hypothesis (HL) is satisfied. But we need only the weaker hypothesis (H) to get Lemmas 4 and 5.

A finite sequence of polynomials $\{k_1, k_2, \ldots, k_n\} \subset \mathbb{F}_q[z]$ is said to be *admissible* if it satisfies the *admissibility condition*

$$\deg k_1 \geqslant 1,$$

$$\deg k_{j+1} \geqslant 2\deg k_j + 1 + \deg r_j(k_j) - \deg s_j(k_j) \quad (1 \leqslant j \leqslant n - 1).$$

We similarly define the admissibility for an infinite sequence of polynomials. It is clear that the digital sequences $\{a_n(x)\}_{n \geqslant 1}$ of the formal Laurent series are just all possible admissible sequences (Theorem A).

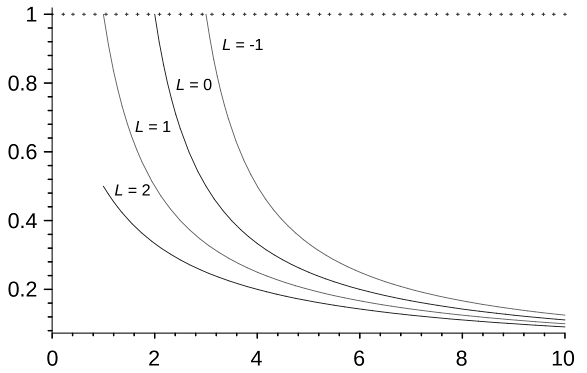

Fig. 1. The curve of dim $E(\tau,\xi)$ for $L = 2, 1, 0, -1$ as function of τ, which is independent of ξ.

Lemma 4. *Let $x \in I$ whose Oppenheim expansion is*

$$x = \sum_{n=1}^{\infty} c_n(x), \quad c_n(x) = \frac{r_1(a_1(x)) \cdots r_{n-1}(a_{n-1}(x))}{s_1(a_1(x)) \cdots s_{n-1}(a_{n-1}(x))} \frac{1}{a_n(x)}.$$

We have $||c_{n+1}(x)|| < ||c_n(x)||$ for all $n \geqslant 1$.

Proof. Notice that

$$v(c_n(x)) = \sum_{j=1}^{n-1} (\deg s_j(a_j(x)) - \deg r_j(a_j(x))) + \deg a_n(x).$$

The difference $v(c_{n+1}(x)) - v(c_n(x))$ is equal to

$$\deg s_n(a_n(x)) - \deg r_n(a_n(x)) + \deg a_{n+1}(x) - \deg a_n(x),$$

which is strictly positive by the admissibility condition (3). □

Lemma 5. *Suppose that $\{k_1, k_2, \ldots, k_n\} \subset \mathbb{F}_q[z]$ $(n \geqslant 1)$ is an admissible sequence. Then the set*

$$\{x \in I\colon\ a_1(x) = k_1, a_2(x) = k_2, \ldots, a_n(x) = k_n\}$$

is equal to the disc $B(C_n, D_n)$ with center

$$C_n := \frac{1}{k_1} + \sum_{j=2}^{n} \frac{r_1(k_1) \cdots r_{j-1}(k_{j-1})}{s_1(k_1) \cdots s_{j-1}(k_{j-1})} \frac{1}{k_j},$$

and diameter

$$D_n := q^{\sum_{j=1}^{n-1} (\deg r_j(k_j) - \deg s_j(k_j)) - 2 \deg k_n - 1}.$$

Proof. For any $1 \leqslant m \leqslant n$, we define C_m and D_m in the same way as C_n and D_n.

For any $x \in I$ such that $a_1(x) = k_1, a_2(x) = k_2, \ldots, a_n(x) = k_n$, we develop it into its Oppenheim expansion (see Theorem A)

$$x = C_n + \sum_{j=n+1}^{\infty} c_j(x),$$

where the functions $c_j(x)$ are defined as in Lemma 4. By Lemma 4, we get

$$||x - C_n|| = ||c_{n+1}(x)|| = q^{\sum_{j=1}^{n} (\deg r_j(k_j) - \deg s_j(k_j)) - \deg a_{n+1}(x)}.$$

By the admissibility condition (3),

$$\deg r_n(k_n) - \deg s_n(k_n) - \deg a_{n+1}(x) \leqslant -2 \deg k_n - 1.$$

It follow that $x \in B(C_n, D_n)$. Thus we get

$$\{x \in I\colon\ a_1(x) = k_1, a_2(x) = k_2, \ldots, a_n(x) = k_n\} \subset B(C_n, D_n).$$

Prove now the inverse inclusion. Let $y \in B(c_n, D_n)$. According to Theorem A, write

$$y = \sum_{j=1}^{\infty} c_j(y).$$

We are going to show that $a_j(y) = k_j$ for $1 \leqslant j \leqslant n$. Let us first prove $a_1(y) = k_1$. Suppose $a_1(y) \neq k_1$. There are two cases.

Case I: $\deg a_1(y) \neq \deg k_1$. In this case, we have

$$\left|\left|\frac{1}{a_1(y)} - \frac{1}{k_1}\right|\right| = \max\left(\left|\left|\frac{1}{a_1(y)}\right|\right|, \left|\left|\frac{1}{k_1}\right|\right|\right).$$

By Lemma 4, we have

$$\left|\left|\frac{1}{a_1(y)}\right|\right| > ||c_j(y)|| \quad (j \geqslant 2); \qquad \left|\left|\frac{1}{k_1}\right|\right| > ||c_j|| \quad (2 \leqslant j \leqslant n).$$

Therefore

$$\max\left(\left|\left|\frac{1}{a_1(y)}\right|\right|, \left|\left|\frac{1}{k_1}\right|\right|\right) > \max\left(\max_{2 \leqslant j \leqslant n} \max(||c_j(y)||,\ ||c_j||),\ \sup_{j \geqslant n+1} ||c_j(y)||\right).$$

It follows that

$$||y - C_n|| = \left|\left|\frac{1}{a_1(y)} - \frac{1}{k_1}\right|\right| = \max\left(\left|\left|\frac{1}{a_1(y)}\right|\right|, \left|\left|\frac{1}{k_1}\right|\right|\right) \geqslant \left|\left|\frac{1}{k_1}\right|\right| = q^{-\deg k_1}.$$

By the admissibility condition (3), we may check that $\log_q D_m$ is decreasing so that

$$D_n \leqslant D_1 = q^{-2 \deg k_1 - 1} < q^{-\deg k_1} \leqslant ||y - C_n||,$$

which contradicts the fact $y \in B(C_n, D_n)$.

Case II: $\deg a_1(y) = \deg k_1$. In this case, using (3), we can prove by induction on j that

$$\left|\left|\frac{1}{a_1(y)}\right|\right|^2 > ||c_j(y)|| \quad (j \geqslant 2); \qquad \left|\left|\frac{1}{k_1}\right|\right|^2 > ||c_j|| \quad (2 \leqslant j \leqslant n).$$

Thus

$$\left\|\frac{1}{a_1(y)}-\frac{1}{k_1}\right\| = \left\|\frac{k_1-a_1(y)}{k_1a_1(y)}\right\| \geqslant \left\|\frac{1}{k_1}\right\|^2 = \left\|\frac{1}{a_1(y)}\right\|^2$$
$$> \max\left(\max_{2\leqslant j\leqslant n} \max(\|c_j(y)\|, \|c_j\|), \sup_{j\geqslant n+1} \|c_j(y)\|\right).$$

Therefore

$$\|y-C_n\| = \left\|\frac{1}{a_1(y)}-\frac{1}{k_1}\right\| \geqslant \left\|\frac{1}{k_1}\right\|^2 = q^{-2\deg k_1} > D_1 \geqslant D_n.$$

Hence we have proved $a_1(y)=k_1$. In the same way, we can show successively $a_2(y)=k_2, \dots, a_n(y)=k_n$. □

We finishing this section by stating the mass distribution principle (see [2, Proposition 4.2]) that will be used several times.

Lemma 6. *Let $E\subset I$ be a Borel set and μ be a probability measure with $\mu(E)>0$. If there exist constants $c>0$ and $\delta>0$ such that*

$$\mu(D)\leqslant c(\operatorname{diam} D)^s$$

for all disc D with diameter $\operatorname{diam} D\leqslant\delta$. *Then*

$$\dim E\geqslant s.$$

3. Proof: case $L\leqslant 0$

For any $\tau>2-L$ and $a>0$, choose a sufficient large integer $j_0\geqslant 1$ such that

$$(2-L)^{j_0+1}+1\leqslant[\tau^{j_0+1}a], \quad \frac{2}{a(\tau-(2-L))}\leqslant\tau^{j_0} \tag{4}$$

and define a sequence of integers as follows:

$$q_1=1; \quad q_j=(2-L)q_{j-1}+1 \ (2\leqslant j\leqslant j_0); \quad q_j=[\tau^j a] \ (j>j_0).$$

By condition (4), it may be checked that

$$q_{j+1}\geqslant(2-L)q_j+1 \quad (\forall j\geqslant 1). \tag{5}$$

Then define

$$F_{\tau,a}=\{x\in I\colon \ \deg a_j(x)=q_j \ \forall j\geqslant 1\}.$$

Proposition 7. *Suppose $L\leqslant 0$. Then for any $\tau>2-L$ and $a>0$, we have*

$$\dim F_{\tau,a}=\frac{1}{L+\tau-1}.$$

Proof. Let us first estimate the dimension from below by using the mass distribution principle. For any $n \geqslant 1$, let $\mathbb{F}_q^{(q_n)}[z]$ denote the set of the polynomials in $\mathbb{F}_q[z]$ with degree q_n. Such polynomials are of the form

$$x = \sum_{k=0}^{q_n} c_k z^k \quad (c_k \in \mathbb{F}_q, \ c_{q_n} \neq 0).$$

For $b_1 \in \mathbb{F}_q^{(q_1)}[z], b_2 \in \mathbb{F}_q^{(q_2)}[z], \ldots, b_n \in \mathbb{F}_q^{(q_n)}[z]$ $(n \geqslant 1)$, define

$$J(b_1, b_2, \ldots, b_n) = \{x \in I\colon \ a_1(x) = b_1, a_2(x) = b_2, \ldots, a_n(x) = b_n\}.$$

We call $J(b_1, b_2, \ldots, b_n)$ an *n-digital cylinder*.

Remark that the sequence $(b_1, b_2, \ldots, b_n)$ is admissible because of (5). Let

$$M(b_1, b_2, \ldots, b_n) = \bigcup_{b_{n+1} \in \mathbb{F}_q^{(q_{n+1})}[z]} J(b_1, b_2, \ldots, b_n, b_{n+1}).$$

Such a set $M(b_1, b_2, \ldots, b_n)$ is call a *n-basic cylinder*. It is easy to see

$$F_{\tau,a} = \bigcap_{n=1}^{\infty} E_n,$$

where

$$E_n = \{x \in I\colon \ \deg a_j(x) = q_j, \ 1 \leqslant j \leqslant n\} = \bigcup_{b_1, b_2, \ldots, b_n} M(b_1, b_2, \ldots, b_n).$$

The n-basic cylinders have the following properties:

(i) The diameter $|M(b_1, b_2, \ldots, b_n)| = q^{-L \sum_{j=1}^{n} q_j - q_{n+1}}$.
(ii) All $M(b_1, b_2, \ldots, b_n)$ are disjoint.
(iii) The number of n-basic cylinders is equal to $(q-1)^n q^{\sum_{j=1}^{n} q_j}$.

In fact, property (iii) is obvious and property (ii) is clear because

$$M(b_1, b_2, \ldots, b_n) \subset J(b_1, b_2, \ldots, b_n)$$

and that $J(b_1, b_2, \ldots, b_n)$'s are disjoint discs. Prove now property (i). Let x and y be two points in $M(b_1, b_2, \ldots, b_n)$. Then there exist $b_{n+1}, b'_{n+1} \in \mathbb{F}_q^{(q_{n+1})}[z]$ such that

$$x \in J(b_1, b_2, \ldots, b_{n+1}), \ \ y \in J(b_1, b_2, \ldots, b'_{n+1}).$$

We distinguish two cases.

Case I: $b_{n+1} = b'_{n+1}$: In this case, both x and y belong to the same $(n+1)$-digital cylinder $J(b_1, b_2, \ldots, b_{n+1})$ so that, by Lemma 5, we have

$$||x - y|| \leqslant q^{-L \sum_{j=1}^{n} q_j - 2q_{n+1} - 1} < q^{-L \sum_{j=1}^{n} q_j - q_{n+1}}. \tag{6}$$

Case II: $b_{n+1} \neq b'_{n+1}$: In this case, we have

$$\begin{aligned}||x-y|| &= \left|\left|\frac{r_1(b_1)\cdots r_n(b_n)}{s_1(b_1)\cdots s_n(b_n)}\left(\frac{1}{b_{n+1}}-\frac{1}{b'_{n+1}}\right)\right|\right| \\ &= q^{-L\sum_{j=1}^{n} q_j}\, q^{-2q_{n+1}+\deg(b_{n+1}-b'_{n+1})} \\ &\leqslant q^{-L\sum_{j=1}^{n} q_j - q_{n+1}} \end{aligned} \tag{7}$$

because of $\deg(b_{n+1}-b'_{n+1}) \leqslant q_{n+1}$.

Notice that the last inequality becomes equality if $\deg(b_{n+1}-b'_{n+1}) = q_{n+1}$. Then (i) follows from (6) and (7).

We define a probability measure μ on the compact set $F_{\tau,a}$ by

$$\mu(M(b_1,b_2,\ldots,b_n)) = (q-1)^{-n} q^{-\sum_{j=1}^{n} q_j}.$$

(It does define a measure by the Carathéodory extension theorem).

For any $\varepsilon > 0$, there is an integer $n_0 = n_0(\varepsilon)$ such that

$$\frac{\sum_{j=1}^{n} q_j}{L\sum_{j=1}^{n} q_j + q_{n+1}} > \frac{1}{L+\tau-1} - \varepsilon \quad (\forall n \geqslant n_0), \tag{8}$$

$$\frac{(L-1)\sum_{j=1}^{n-1} q_j + q_n}{L\sum_{j=1}^{n-1} q_j + q_n} < \frac{L+\tau-2}{L+\tau-1} + \varepsilon \quad (\forall n \geqslant n_0). \tag{9}$$

For any $m > \sum_{j=1}^{n_0-1} q_j + q_{n_0}$, let $n \geqslant n_0$ be the integer such that

$$q^{-L\sum_{j=1}^{n} q_j - q_{n+1}} \leqslant q^{-m} < q^{-L\sum_{j=1}^{n-1} q_j - q_n}.$$

One of the following two situations will occur:

$$q^{-L\sum_{j=1}^{n} q_j - q_{n+1}} \leqslant q^{-m} < q^{-L\sum_{j=1}^{n-1} q_j - 2q_n - 1}, \text{(S1)}$$

$$q^{-L\sum_{j=1}^{n-1} q_j - 2q_n - 1} \leqslant q^{-m} < q^{-L\sum_{j=1}^{n-1} q_j - q_n}. \text{(S2)}$$

Suppose (S1): For any $x \in F_{\tau,a}$, if the disc $B(x,q^{-m}) := \{y \in I\colon ||y-x|| \leqslant q^{-m}\}$ intersects some n-basic cylinder $M(b_1,b_2,\ldots,b_n)$, then we have

$$B(x,q^{-m}) \subset J(b_1,b_2,\ldots,b_n). \tag{10}$$

Indeed, $M(b_1,b_2,\ldots,b_n) \bigcap B(x,q^{-m}) \neq \emptyset$ implies that $B(x,q^{-m})$ intersects $J(b_1,b_2,\ldots,b_n)$. However, both $J(b_1,b_2,\ldots,b_n)$ and $B(x,q^{-m})$ are discs. It suffices to compare their diameters, which is easy, to get the claimed inclusion. According to (10), the disc $B(x,q^{-m})$ intersects one and exact one n-basic cylinder

$M(a_1(x), a_2(x), \ldots, a_n(x))$. Hence

$$\begin{aligned}\mu(B(x, q^{-m})) &\leqslant \mu(M(a_1(x), a_2(x), \ldots, a_n(x))) \\ &= (q-1)^{-n} q^{-\sum_{j=1}^{n} q_j} \\ &\leqslant (q-1)^{-n} q^{-m\left(\frac{\sum_{j=1}^{n} q_j}{L\sum_{j=1}^{n} q_j + q_{n+1}}\right)} \\ &\leqslant q^{-m\left(\frac{1}{L+\tau-1}-\varepsilon\right)} \end{aligned} \tag{11}$$

((8) was used to get the last inequality).

Suppose (S2): We claim that for any $x \in F_{\tau,a}$, the number of n-basic cylinders which intersect $B(x, q^{-m})$ is bounded by

$$q^{L\sum_{j=1}^{n-1} q_j + 2q_n - m + 1}.$$

In fact, assume that $M(b_1, b_2, \ldots, b_n) \bigcap B(x, q^{-m}) \neq \emptyset$. Then $B(x, q^{-m})$ intersects $J(b_1, b_2, \ldots, b_n)$. Recall that

$$|J(b_1, b_2, \ldots, b_n)| = q^{-L\sum_{j=1}^{n-1} q_j - 2q_n - 1} \leqslant q^{-m},$$

$$|J(b_1, b_2, \ldots, b_{n-1})| = q^{-L\sum_{j=1}^{n-2} q_j - 2q_{n-1} - 1} \geqslant q^{-L\sum_{j=1}^{n-1} q_j - q_n} > q^{-m}$$

where the next to the last inequality is assumed by (5). It follows that

$$J(b_1, b_2, \ldots, b_n) \subset B(x, q^{-m}) \subset J(b_1, b_2, \ldots, b_{n-1}). \tag{12}$$

By the second inclusion in (12), we get $b_1 = a_1(x), \ldots, b_{n-1} = a_{n-1}(x)$. If $b_n \neq a_n(x)$, by the first inclusion in (12), we have

$$\left\|\frac{r_1(b_1)\cdots r_{n-1}(b_{n-1})}{s_1(b_1)\cdots s_{n-1}(b_{n-1})}\frac{1}{b_n} - \frac{r_1(b_1)\cdots r_{n-1}(b_{n-1})}{s_1(b_1)\cdots s_{n-1}(b_{n-1})}\frac{1}{a_n(x)}\right\| \leqslant q^{-m},$$

i.e.

$$\deg(b_n - a_n(x)) \leqslant L\sum_{j=1}^{n-1} q_j + 2q_n - m.$$

Write

$$b_n = c_0 + c_1 z + \cdots + c_{q_n} z^{q_n} \quad (c_{q_n} \neq 0),$$

$$a_n(x) = c'_0 + c'_1 z + \cdots + c'_{q_n} z^{q_n} \quad (c'_{q_n} \neq 0).$$

We must have

$$c_k = c'_k \quad \text{for all } k > k_0 = L\sum_{j=1}^{n-1} q_j + 2q_n - m.$$

So, the number of choices for b_n is bounded by the number of choices for $c_0, \dots, c_{k_0}$, which is bounded by

$$q^{k_0+1} = q^{L \sum_{j=1}^{n-1} q_j + 2q_n - m + 1}.$$

Thus the claim is proved. Therefore

$$\begin{aligned}\mu(B(x, q^{-m})) &\leqslant q^{k_0+1} (q-1)^{-n} q^{-\sum_{j=1}^{n} q_j} \\ &= (q-1)^{-n} q\, q^{(L-1) \sum_{j=1}^{n-1} q_j + q_n - m} \\ &\leqslant q\, q^{-m(\frac{1}{L+\tau-1} - \varepsilon)}. \end{aligned} \tag{13}$$

Let us check the last inequality, which is implied by

$$(L-1) \sum_{j=1}^{n-1} q_j + q_n - m \leqslant -m\left(\frac{1}{L+\tau-1} - \varepsilon\right)$$

or equivalently

$$(L-1) \sum_{j=1}^{n-1} q_j + q_n \leqslant m\left(\frac{L+\tau-2}{L+\tau-1} + \varepsilon\right). \tag{14}$$

However since

$$m \geqslant L \sum_{j=1}^{n-1} q_j + q_n,$$

(14) is implied by condition (9).

By (11) and (13) and Lemma 6, we get

$$\dim F_{\tau,a} \geqslant \frac{1}{L+\tau-1}.$$

The inverse inequality may be obtained by using the fact that for any $n \geqslant 1$, n-basic cylinders form a cover for $F_{\tau,a}$. Then by (i) and (iii), we have

$$\dim F_{\tau,a} \leqslant \lim_{n\to\infty} \frac{n \log(q-1) + (\sum_{j=1}^{n} q_j) \log q}{(L \sum_{j=1}^{n} q_j + q_{n+1}) \log q} = \frac{1}{L+\tau-1}.$$

Proof of Theorem 1. Let $E(\tau, \xi)$ be the set of $x \in I$ such that $\frac{1}{\tau^n} \log_q ||x - \omega_n(x)|| \to -\xi$. By Lemma 4, we have

$$||x - \omega_n(x)|| = q^{-L \sum_{j=1}^{n} \deg a_j(x) - \deg a_{n+1}(x)}.$$

We can write

$$L \sum_{j=1}^{n} \deg a_j(x) + \deg a_{n+1}(x) = \sum_{j=1}^{n} (\deg a_{j+1}(x) - (1-L) \deg a_j(x)) + \deg a_1(x).$$

Thus

$$E(\tau,\xi) = \left\{ x \in I\colon \frac{1}{\tau^n} \sum_{j=1}^{n} (\deg a_{j+1}(x) - (1-L)\deg a_j(x)) \to \xi \right\}.$$

It is easy to check that

$$F_{\tau,a} \subset E(\tau,\xi) \quad \text{with} \quad a = \frac{\xi(\tau-1)}{\tau(L+\tau-1)} > 0$$

where $F_{\tau,a}$ is defined in Proposition 7. By Proposition 7, we get

$$\dim E(\tau,\xi) \geqslant \dim F_{\tau,a} = \frac{1}{L+\tau-1}.$$

Prove now the inverse inequality. Denote

$$\sigma_n(x) = \sum_{j=1}^{n} (\deg a_{j+1}(x) - (1-L)\deg a_j(x)).$$

For any $\varepsilon > 0$, we have

$$E(\tau,\xi) \subset \bigcup_{N=1}^{\infty} G_N,$$

where

$$G_N = \bigcap_{n=N}^{\infty} \{x \in I\colon \tau^n(\xi-\varepsilon) < \sigma_n(x) < \tau^n(\xi+\varepsilon)\}.$$

By the σ-stability of the Hausdorff dimension, we have only to show that

$$\dim G_N \leqslant \frac{c_2(\varepsilon)}{c_1(\varepsilon)(L+\tau-1)} \quad (\forall N \geqslant 1,\ \forall \varepsilon > 0),$$

where

$$c_1(\varepsilon) = (\tau-1)\xi - (\tau+1)\varepsilon, \quad c_2(\varepsilon) = (\tau-1)\xi + (\tau+1)\varepsilon.$$

In the sequel, we only give a proof of $N=1$ (the case $N>1$ may be treated in the same way).

Since $\sigma_n(x) - \sigma_{n-1}(x) = \deg a_{n+1}(x) - (1-L)\deg a_n(x)$, G_1 is contained in

$$H := \bigcap_{n=1}^{\infty} \{x \in I\colon \tau^{n-1}c_1(\varepsilon) \leqslant \deg a_{n+1}(x) - (1-L)\deg a_n(x) \leqslant \tau^{n-1}c_2(\varepsilon)\}.$$

For any $x \in H$ and any $n \geqslant 1$, we deduce by induction that

$$\begin{aligned}\deg a_n(x) &\leqslant c_2(\varepsilon)\tau^{n-1} + (1-L)\deg a_n(x)\\ &\leqslant c_2(\varepsilon)(\tau^{n-1} + (1-L)\tau^{n-2}) + (1-L)^2 \deg a_{n-1}(x)\\ &\vdots\\ &\leqslant c_2(\varepsilon)\tau^{n-1}\sum_{j=0}^{n-1}\left(\frac{1-L}{\tau}\right)^j + (1-L)^n \deg a_1(x)\\ &\leqslant c_2(\varepsilon)\frac{\tau^n}{L+\tau-1} + (1-L)^n \deg a_1(x),\end{aligned}$$

and that

$$\begin{aligned}\deg a_n(x) &\geqslant c_1(\varepsilon)\tau^{n-1}\sum_{j=0}^{n-1}\left(\frac{1-L}{\tau}\right)^j - (1-L)^n \deg a_1(x)\\ &= c_1(\varepsilon)\frac{\tau^n}{L+\tau-1}\left(1-\left(\frac{1-L}{\tau}\right)^n\right) - (1-L)^n \deg a_1(x).\end{aligned}$$

Let

$$\begin{aligned}S(n,k) &= c_1(\varepsilon)\frac{\tau^n}{L+\tau-1}\left(1-\left(\frac{1-L}{\tau}\right)^n\right) - (1-L)^n k,\\ T(n,k) &= c_2(\varepsilon)\frac{\tau^n}{L+\tau-1} + (1-L)^n k.\end{aligned}$$

The proceeding estimates lead to

$$G_1 \subset H \subset W := \bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty} W_{k,n},$$

where $W_{k,n} = \{\deg a_1(x) = k; S(j,k) \leqslant \deg a_{j+1}(x) \leqslant T(j,k),\ 1 \leqslant j \leqslant n-1\}$. Again by the σ-stability of the Hausdorff dimension, we only need to show

$$\dim W_{n,k} \leqslant \frac{c_2(\varepsilon)}{c_1(\varepsilon)(L+\tau-1)} \quad (\forall k \geqslant 1, \forall \varepsilon > 0).$$

Define $I(b_1, b_2, \ldots, b_n) = \{x \in I\colon\ a_1(x) = b_1, \ldots, a_n(x) = b_n\}$ for any finite sequence $b_1, \ldots, b_n$ in $\mathbb{F}_q[z]$. Observe that

$$W_{n,k} = \bigcup_{b_1,\ldots,b_n} I(b_1, b_2, \ldots, b_n),$$

where the union is taken over all polynomials $b_1, b_2, \ldots, b_n$ such that

$$\deg b_1 = k;\ S(j,k) \leqslant \deg b_{j+1} \leqslant T(j,k), \quad 1 \leqslant j \leqslant n-1. \tag{15}$$

Let

$$M(b_1, b_2, \ldots, b_n) = \bigcup_{S(n,k) \leqslant \deg b_{n+1} \leqslant T(n,k)} I(b_1, b_2, \ldots, b_n, b_{n+1})$$

and

$$W^*_{k,n} = \bigcup_{b_1, \ldots, b_n} M(b_1, b_2, \ldots, b_n),$$

where the union is taken over all sequences $b_1, b_2, \ldots, b_n$ satisfying (15). Then we can write

$$\bigcap_{n=1}^{\infty} W_{k,n} = \bigcap_{n=1}^{\infty} W^*_{k,n}.$$

It follows that for any $n \geqslant 1$, all $M(b_1, b_2, \ldots, b_n)$ with $b_1, b_2, \ldots, b_n$ satisfying (15) is a cover of $\bigcap_{n=1}^{\infty} W_{k,n}$. We will use this cover to give a upper bound of the Hausdorff dimension of $\bigcap_{n=1}^{\infty} W_{k,n}$.

Actually, all $I(b_1, b_2, \ldots, b_n)$ with $b_1, b_2, \ldots, b_n$ satisfying (15) is also a cover of $\bigcap_{n=1}^{\infty} W_{k,n}$. We could use this cover to estimate the dimension, which is however not effective.

Let us first estimate the diameter of $M(b_1, b_2, \ldots, b_n)$ and the number of all $M(b_1, b_2, \ldots, b_n)$ in the cover.

By using (15), we can prove, in the same way as we prove (6) and (7), that

$$|M(b_1, b_2, \ldots, b_n)| \leqslant q^{-LK - L\sum_{j=1}^{n-1} S(j,k) - S(n,k)} := R_n.$$

The number of $M(b_1, b_2, \ldots, b_n)$'s is equal to

$$N_n := (q-1)q^k \cdot \prod_{j=1}^{n-1} \left(\sum_{S(j,k) \leqslant i \leqslant T(j,k)} (q-1)q^i \right) \leqslant (q-1)q^{n+k} q^{\sum_{j=1}^{n-1} T(j,k)}.$$

So

$$\begin{aligned}
\dim \bigcap_{n=1}^{\infty} W^*_{k,n} &\leqslant \liminf_{n \to \infty} \frac{\log N_n}{-\log R_n} \\
&= \lim_{n \to \infty} \frac{\sum_{j=1}^{n-1} T(j,k)}{L \sum_{j=1}^{n-1} S(j,k) + S(n,k)} \\
&= \lim_{n \to \infty} \frac{T(n,k)}{LS(n,k) + S(n+1,k) - S(n,k)} \\
&= \frac{c_2(\varepsilon)}{c_1(\varepsilon)(L + \tau - 1)},
\end{aligned}$$

where we have used the Toeplitz Lemma to obtain the next to the last equality.

4. Proof: case $L = 2$

Let $A(\beta, \alpha)$ be the set of $x \in I$ such that $\frac{1}{n^\beta} \log_q ||x - \omega(x)|| \to -\alpha$. By Lemma 4, we have

$$||x - \omega_n(x)|| = q^{-2\sum_{j=1}^n \deg a_j(x) - \deg a_{n+1}(x)}.$$

We construct a subset $F_{\beta,a}$ of I as follows:

$$F_{\beta,a} = \{x \in I\colon \ \deg a_n(x) = 1 + [n^{\beta-1}a] \ \forall n \geqslant 1\}.$$

It may be checked that

$$F_{\beta,a} \subset A(\beta, \alpha) \quad \text{for } a = \frac{\alpha\beta}{2}.$$

By similar arguments in the proof of Proposition 7 (the actual case is simpler because we can use directly $J(b_1, \ldots, b_n)$'s instead of $M(b_1, \ldots, b_n)$'s), we can show that

$$\dim F_{\beta,a} = \tfrac{1}{2}.$$

Thus we get $\dim A(\beta, \alpha) \geqslant \frac{1}{2}$.

In order to get the upper bound, first notice that when $L = 2$, every sequence $\{b_1, b_2, \ldots, b_n\} \subset \mathbb{F}_q[z]$ such that $\deg b_j \geqslant 1$ $(1 \leqslant j \leqslant n)$ is admissible. Recall that

$$I(b_1, b_2, \ldots, b_n) = \{x \in I\colon \ a_1(x) = b_1, \ldots, a_n(x) = b_n\}.$$

We construct a family of measure μ_t on I for $t > \log q$ as follows:

$$\mu_t(I(b_1, b_2, \ldots, b_n)) = \exp\left(-t \sum_{j=1}^n \deg b_j - nP(t)\right),$$

where $P(t) = \log(q(q-1)) - \log(e^t - q)$.

Observe that, for any $\varepsilon > 0$, we have

$$A(\beta, \alpha) \subset \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty A_n(\beta, \alpha), \tag{16}$$

where

$$A_n(\beta, \alpha) = \left\{x \in I\colon \ n^\beta(\alpha - \varepsilon) \leqslant 2\sum_{j=1}^n \deg a_j(x) + \deg a_{n+1}(x) \leqslant n^\beta(\alpha + \varepsilon)\right\}.$$

Let $\mathscr{I}(n, \alpha, \varepsilon)$ be the family of all $J(b_1, b_2, \ldots, b_n)$ such that

$$\deg b_j \geqslant 1, \ (1 \leqslant j \leqslant n); \quad n^\beta(\alpha - \varepsilon) \leqslant 2\sum_{j=1}^n \deg b_j + \deg b_{n+1} \leqslant n^\beta(\alpha + \varepsilon).$$

For $N \geqslant 1$, we select all those discs in $\bigcup_{n=N}^\infty \mathscr{I}(n, \alpha, \varepsilon)$ which are maximal ($J \in \bigcup_{n=N}^\infty \mathscr{I}(n, \alpha, \varepsilon)$ is maximal if there is no $J' \in \bigcup_{n=N}^\infty \mathscr{I}(n, \alpha, \varepsilon)$ such that $J \subset J'$ and $J \neq J'$). We denote by $\mathscr{J}(N, \alpha, \varepsilon)$ the set of all maximal J in $\bigcup_{n=N}^\infty \mathscr{I}(n, \alpha, \varepsilon)$.

From (16), we see that for any $N \geqslant 1$, $\mathscr{J}(N, \alpha, \varepsilon)$ is a cover of $A(\beta, \alpha)$. Let $I(b_1, b_2, \dots, b_n) \in \mathscr{J}(N, \alpha, \varepsilon)$, we have

$$\mu_\beta(I(b_1, b_2, \dots, b_n)) = \exp\left(-t \sum_{j=1}^{n} \deg b_j - nP(t)\right)$$
$$\geqslant \exp\Big(-\frac{t}{2} \cdot n^\beta(\alpha + \varepsilon) - nP(t)\Big)$$

and

$$|I(b_1, b_2, \dots, b_n)| = q^{-2\sum_{j=1}^{n} \deg b_j - 1} \leqslant q^{-2\sum_{j=1}^{n-1} \deg b_j - \deg b_n} \leqslant q^{-(n-1)^\beta(\alpha - \varepsilon)}.$$

Let $\delta = \frac{3\varepsilon t}{\alpha - \varepsilon}$. Then for sufficient large n, we have

$$\frac{t}{2} n^\beta(\alpha + \varepsilon) - nP(t) - \frac{t + \delta}{2}(n-1)^\beta(\alpha - \varepsilon) \leqslant 0.$$

So,

$$\sum_{I(b_1, b_2, \dots, b_n) \in \mathscr{J}(N, \alpha, \varepsilon)} |I(b_1, b_2, \dots, b_n)|^{\frac{t+\delta}{2\log q}}$$
$$\sum_{I(b_1, b_2, \dots, b_n) \in \mathscr{J}(N, \alpha, \varepsilon)} \exp\Big(-\frac{t}{2} \cdot n^\beta(\alpha + \varepsilon) - nP(t)\Big) \leqslant 1.$$

It follows that

$$\dim A(\beta, \alpha) \leqslant \frac{t + \delta}{2 \log q}.$$

Letting $t \to \log q$ then $\varepsilon \to 0$ (i.e. $\delta \to 0$), we get $\dim A(\beta, \alpha) \leqslant \frac{1}{2}$.

Prove now the second part of Theorem 3. It is the same proof as that of Theorem 1. We just point out the small differences. Let $E(\tau, \xi)$ be the set in question. We define

$$F_{\tau,a} = \{x \in I\colon\ \deg a_n(x) = [\tau^n a] + 1 \ \forall n \geqslant 1\}$$

for any $a > 0$. We have

$$F_{\tau,a} \subset E(\tau, \xi) \quad \text{for } a = \frac{\xi(\tau - 1)}{\tau(\tau + 1)}.$$

By estimating the dimension of $F_{\tau,a}$, we get a lower bound of $E(\tau, \xi)$. The upper bound of $E(\tau, \xi)$ may be obtained as before (see the proof of Theorem 1).

5. Proof: case $L = 1$

Let $A(\beta, \alpha)$ be the set of $x \in I$ such that $\frac{1}{n^\beta} \log_q ||x - \omega(x)|| \to -\alpha$. By Lemma 4, we have

$$||x - \omega_n(x)|| q^{-\sum_{j=1}^{n} \deg a_j(x) - \deg a_{n+1}(x)}.$$

For any $a > 0$, choose $j_0 \geqslant 1$ such that

$$[n^{\beta-1}a] \geqslant j_0 + 1, [(n+1)^{\beta-1}a] - [n^{\beta-1}a] \geqslant 1 \quad (\forall j > j_0).$$

Define

$$q_j = j \ (1 \leqslant j \leqslant j_0), \quad q_j = [j^{\beta-1}a] \quad (\forall j > j_0)$$

and

$$F_{\beta,a} = \{x \in I: \ \deg a_j(x) = q_j, \ \forall j \geqslant 1\}.$$

We have

$$F_{\beta,a} \subset A(\beta, \alpha) \quad \text{with} \quad a = \alpha\beta.$$

Also we may follow the proof of Proposition 7 to show that

$$\dim F_{\beta,a} = 1 \quad (\forall a > 0),$$

thus we have $\dim A(\beta, \alpha) = 1$.

The proof of the result concerning the exponential degree is the same as that of Theorem 3. This time, in order to get the lower bound of $\dim E(\tau, \xi)$, we need

$$F_{\beta,a} = \{x \in I: \ \deg a_j(x) = q_j, \ \forall j \geqslant 1\} \quad (a > 0),$$

where

$$q_j = j \ (1 \leqslant j \leqslant j_0); \quad q_j [\tau^j a] \quad (j > j_0).$$

The integer j_0 here is choosen so that

$$[\tau^j a] \geqslant j_0 + 1, \quad [\tau^{j+1} a] - [\tau^j a] \geqslant 1 \quad (\forall j > j_0).$$

We have

$$F_{\tau,a} \subset E(\tau, \xi) \quad \text{for } a = \frac{\xi(\tau - 1)}{\tau^2}.$$

The upper bound of $\dim E(\tau, \xi)$ is obtained in exactly the same way as before (see the proof of Theorem 1).

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